A necessary and sufficient criterion for multipartite separable states

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Abstract

We present a necessary and sufficient condition for the separability of multipartite quantum states, this criterion also tells us how to write a multipartite separable state as a convex sum of separable pure states. To work out this criterion, we need to solve a set of equations, actually it is easy to solve these equations analytically if the density matrix of the given quantum state has few nonzero eigenvalues.

Ever since it was first noted by Einstein-Podolsky-Rosen (EPR) [1] and Schrödinger [2], entanglement has played an important role in quantum information theory. Quantum entanglement provides strong tests of quantum nonlocality [3,4], and it is also a useful resource for various kinds of quantum information processing, including teleportation [5,6], entanglement swapping [7,8], cryptographic key distribution [9], quantum error correction [10] and quantum computation [11].

A multipartite quantum state is called separable if it can be written as a convex sum of product states belonging to different parties, otherwise it is called entangled. It is important to know whether a given multipartite quantum state is separable or entangled.

So far, there have been many ingenious separability criteria. Since a separable state always satisfies Bell's inequalities, the latter represent a necessary condition for separability [12], but generally they are not sufficient. Peres [13] discovered another simple necessary condition for separability, a partial

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transposition of a bipartite quantum state ρ_{AB} with respect to a subsystem A (or B) must be positive if ρ_{AB} is separable. Peres' criterion has been shown by Horodecki et al. [14] to be strong enough to guarantee separability for bipartite systems of dimension 2×2 or 2×3, but, for other cases it is not a sufficient one. It has been proved by Horodecki et al. [14] that a necessary and sufficient condition for separability of bipartite mixed state is its positivity under all the maps of the form $I \otimes \Lambda$, where Λ is any positive map. This criterion is more important in theory than in practice since it involves the characterization of the set of all positive maps, which is not easy. More recently, Horodecki-Horodecki [15] and Cerf-Adami-Gingrich [16] have independently derived a reduction criterion of separability for bipartite quantum states, this criterion is equivalent to Peres' for $2 \times n$ composite systems, and it is not sufficient for separability in general cases. Many interesting separability criteria have been presented recently, such as the rank separability criterion derived by Horodecki et al. [17], which shows that a separable state cannot have the rank of a reduced density matrix greater than the rank of total density matrix, this necessary condition is easy for operation.

Here we introduce a necessary and sufficient condition for the separability of multipartite quantum states, this criterion also gives the expression for a separable state in the form of convex sum of product pure states.

Let there be m subsystems A, B, $\cdots$, M belonging to m different observers Alice, Bob, $\cdots$, Mary, respectively. A m-party quantum state $\rho_{AB\cdots M}$ is called separable iff it can be written as

$$\rho_{AB\cdots M} = \sum_{i=1}^{r} p_i \left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle \left\langle \psi_i^A \psi_i^B \cdots \psi_i^M \right| \tag{1}$$

where $\{|\psi_i^\alpha\rangle \,|i=1,2,\cdots r\}$ is a set of normalized (generally not orthogonal) states of system α (α =A,B,$\cdots$,M), and the probabilities $p_i > 0$, $\sum_i^r p_i = 1$. On the other hand, any given quantum state (no matter it is separable or entangled) $\rho_{AB\cdots M}$ can always be written in the orthogonal representation as

$$\rho_{AB\cdots M} = \sum_{i=1}^{k} \lambda_i \left| \phi_i^{AB\cdots M} \right\rangle \left\langle \phi_i^{AB\cdots M} \right| \tag{2}$$

where $\left| \phi_i^{AB\cdots M} \right\rangle$ is a set of normalized orthogonal eigenstates corresponding to the nonzero eigenvalues $\lambda_i (\lambda_i > 0, \sum_i^k \lambda_i = 1)$. The eigenstates and eigenvalues of $\rho_{AB\cdots M}$ can always be solved by a standard procedure.

Theorem: Let $\left| \phi_i^{AB\cdots M} \right\rangle$ be the eigenstates corresponding to the nonzero eigenvalues λ_i ($i = 1, \cdots k$) for a given m-party quantum state $\rho_{AB\cdots M}$, $\rho_{AB\cdots M}$ is separable if and only if the equations

$$\begin{cases} |\Psi\rangle \equiv \sum_{i=1}^{k} y_i \left| \phi_i^{AB\cdots M} \right\rangle \\ |y_1|^2 + |y_2|^2 + \cdots + |y_k|^2 = 1 \\ \sigma_\alpha \equiv tr_{\overline{\alpha}} (|\Psi\rangle \langle \Psi|) \\ \det (\sigma_\alpha - I) = 0 \quad (\alpha = A, B, \cdots M) \end{cases} \tag{3}$$

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(here α denotes one of the m parties, and $\bar{\alpha}$ denotes the remaining $m-1$ parties) have r different vector solutions $\overrightarrow{y}^{(l)}$ $(l = 1, 2, \cdots, r; r \geq k)$ satisfying the following condition: there exists a set of positive numbers p_i $(\sum_i^r p_i = 1)$, so that

$$
\begin{cases}
M_{ij} \equiv \sqrt{\frac{p_i}{\lambda_j}} y_j^{(i)} \\
M^\dagger M = I_{k \times k}
\end{cases}
\tag{4}
$$

Moreover, if $\rho_{AB \cdots M}$ is separable, it can be written as the following mixture of separable pure states

$$
\rho_{AB \cdots M} = \sum_{i=1}^r p_i \left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle \left\langle \psi_i^A \psi_i^B \cdots \psi_i^M \right|
\tag{5}
$$

where p_i is given by Eq. (4), and $\left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle$ is given by

$$
\left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle = \sum_j y_j^{(i)} \left| \phi_j^{AB \cdots M} \right\rangle
\tag{6}
$$

Here, we say two vectors $\overrightarrow{y}^{(1)}$, $\overrightarrow{y}^{(2)}$ are different if there exists no factor K such that $\overrightarrow{y}^{(2)} = K \cdot \overrightarrow{y}^{(1)}$. Actually we can always choose the first column of the matrix M (i.e., $y_1^{(l)}$) to be non-negative real numbers. And obviously, there are only $m-1$ independent equations among the m equations $\det (\sigma_\alpha - I) = 0$ $(\alpha = A, B, \cdots M)$.

Proof. The theorem can be proved using Hughston-Jozsa-Wootters' result [18] and the properties of the separable pure states, while in the following, we give a simple proof of the theorem directly.

Let us first prove the necessity. Suppose the state $\rho_{AB \cdots M}$ is separable, i.e.,

$$
\rho_{AB \cdots M} = \sum_{i=1}^r p_i \left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle \left\langle \psi_i^A \psi_i^B \cdots \psi_i^M \right|
\tag{7}
$$

set $y_j^{(i)} = \left\langle \phi_j^{AB \cdots M} \middle| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle$. It is obvious that $\overrightarrow{y}^{(i)}$ is the i-th $(i = 1, \cdots, r)$ solution of Eqs. (3), since the state

$$
\left| \Psi^{(i)} \right\rangle \equiv \sum_{j=1}^k y_j^{(i)} \left| \phi_j^{AB \cdots M} \right\rangle = \left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle
\tag{8}
$$

is a separable pure state. Set $M_{ij} \equiv \sqrt{\frac{p_i}{\lambda_j}} y_j^{(i)}$, we can easily to show that $M^\dagger M = I_{k \times k}$ since

$$
\begin{aligned}
\rho_{AB \cdots M} &= \sum_{i=1}^r p_i \left| \psi_i^A \psi_i^B \cdots \psi_i^M \right\rangle \left\langle \psi_i^A \psi_i^B \cdots \psi_i^M \right| \\
&= \sum_{i=1}^r \sum_{j,j'=1}^k p_i y_j^{(i)} y_{j'}^{(i)*} \left| \phi_j^{AB \cdots M} \right\rangle \left\langle \phi_{j'}^{AB \cdots M} \right| \\
&= \sum_{j,j'=1}^k \sqrt{\lambda_j \lambda_{j'}} \left| \phi_j^{AB \cdots M} \right\rangle \left\langle \phi_{j'}^{AB \cdots M} \right| \cdot \sum_{i=1}^r M_{ij} M_{ij'}^*
\end{aligned}
\tag{9}
$$

and

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$$\rho_{AB\cdots M} = \sum_{i=1}^{k} \lambda_i \left|\phi_i^{AB\cdots M}\right\rangle\left\langle\phi_i^{AB\cdots M}\right|$$

$$= \sum_{j,j'=1}^{k} \sqrt{\lambda_j \lambda_{j'}} \left|\phi_j^{AB\cdots M}\right\rangle\left\langle\phi_{j'}^{AB\cdots M}\right| \cdot \delta_{jj'} \tag{10}$$

This completes the proof of necessity.

Next we come to prove the sufficiency. Suppose the Eqs. (3) have already had solutions $\overrightarrow{y}^{(l)}$ $(l = 1, \cdots, r; r \geq k)$ with a proper set of positive numbers p_i satisfying Eqs. (4), then the state $\sum_{i=1}^{k} y_i^{(l)} \left|\phi_i^{AB\cdots M}\right\rangle$ must be a separable pure state since $\det(\sigma_\alpha - I) = 0$ $(\alpha = A, B, \cdots M)$. Set $\sum_{i=1}^{k} y_i^{(l)} \left|\phi_i^{AB\cdots M}\right\rangle = \left|\psi_l^A \psi_l^B \cdots \psi_l^M\right\rangle$. Now we only need to show that

$$\rho_{AB\cdots M} = \sum_{l=1}^{r} p_l \left|\psi_l^A \psi_l^B \cdots \psi_l^M\right\rangle\left\langle\psi_l^A \psi_l^B \cdots \psi_l^M\right| \tag{11}$$

This is obvious since we have

$$\sum_{l=1}^{r} p_l \left|\psi_l^A \psi_l^B \cdots \psi_l^M\right\rangle\left\langle\psi_l^A \psi_l^B \cdots \psi_l^M\right|$$

$$= \sum_{l=1}^{r} p_l \cdot \sum_{i,j=1}^{k} y_i^{(l)} y_j^{(l)*} \left|\phi_i^{AB\cdots M}\right\rangle\left\langle\phi_j^{AB\cdots M}\right|$$

$$= \sum_{i,j=1}^{k} \sqrt{\lambda_i \lambda_j} \left|\phi_i^{AB\cdots M}\right\rangle\left\langle\phi_j^{AB\cdots M}\right| \cdot \sum_{l=1}^{r} M_{li} M_{lj}^*$$

$$= \sum_{i=1}^{k} \lambda_i \left|\phi_i^{AB\cdots M}\right\rangle\left\langle\phi_i^{AB\cdots M}\right|$$

$$= \rho_{AB\cdots M} \tag{12}$$

This completes the proof of sufficiency.

In the theorem, the separability of a given quantum state is determined by solving a set of equations of the vector variable $\overrightarrow{y} = (y_1, y_2, \cdots, y_m)$. If $\rho_{AB\cdots M}$ has few nonzero eigenvalues (i.e., k is small), generally we can get analytic solutions for Eqs. (3). However, if $\rho_{AB\cdots M}$ has many nonzero eigenvalues (i.e., k is great), then it is difficult to work out analytic solutions for the equations in the theorem, only numerical solutions are practical.

Here are some examples.

(1). Let

$$\rho_{AB} = \lambda \left|\phi^+\right\rangle\left\langle\phi^+\right| + (1 - \lambda) \left|\phi^-\right\rangle\left\langle\phi^-\right| \tag{13}$$

As in the theorem, set

$$|\Psi\rangle \equiv y_1 \left|\phi^+\right\rangle + y_2 \left|\phi^-\right\rangle$$

$$= \frac{y_1 + y_2}{\sqrt{2}} |00\rangle + \frac{y_1 - y_2}{\sqrt{2}} |11\rangle \tag{14}$$

Direct calculation yields

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$$\sigma_A = \frac{|y_1 + y_2|^2}{2} |0\rangle \langle 0| + \frac{|y_1 - y_2|^2}{2} |1\rangle \langle 1| \tag{15}$$

From $\det(\sigma_A - I) = 0$, we get

$$y_1 = \pm y_2 \tag{16}$$

Considering the relation $|y_1|^2 + |y_2|^2 = 1$, we have two (and only two) different vector solutions:

$$\overrightarrow{y}^{(1)} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\overrightarrow{y}^{(2)} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

So

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} \sqrt{\frac{p}{\lambda}} & \frac{1}{\sqrt{2}} \sqrt{\frac{p}{1-\lambda}} \\ \frac{1}{\sqrt{2}} \sqrt{\frac{1-p}{\lambda}} & -\frac{1}{\sqrt{2}} \sqrt{\frac{1-p}{1-\lambda}} \end{pmatrix} \tag{17}$$

Let $M^\dagger M = I_{2\times 2}$, we have that

$$\lambda = 1 - \lambda = \frac{1}{2}$$

$$p = 1 - p = \frac{1}{2}$$

And there is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} |\phi^+\rangle \\ \frac{1}{\sqrt{2}} |\phi^-\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} |00\rangle \\ \frac{1}{\sqrt{2}} |11\rangle \end{pmatrix} \tag{18}$$

The conclusion is that for $\lambda = \frac{1}{2}$, ρ_{AB} is separable and $\rho_{AB} = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11|$, and for $\lambda \neq \frac{1}{2}$, ρ_{AB} is entangled.

(2). Let

$$\begin{aligned}
\rho_{AB} = \frac{1}{4} &\left[\frac{1}{\sqrt{2}} \left(|\phi^+\rangle - i |\psi^+\rangle \right) \right] \left[\frac{1}{\sqrt{2}} \left(\langle \phi^+| + i \langle \psi^+| \right) \right] \\
&+ \frac{3}{4} \left[\frac{1}{2\sqrt{3}} \left(-3i |00\rangle + i |11\rangle + |01\rangle + |10\rangle \right) \right] \\
&\cdot \left[\frac{1}{2\sqrt{3}} \left(3i \langle 00| - i \langle 11| + \langle 01| + \langle 10| \right) \right]
\end{aligned} \tag{19}$$

Set

$$\begin{aligned}
|\Psi\rangle \equiv y_1 \cdot &\frac{1}{\sqrt{2}} \left(|\phi^+\rangle - i |\psi^+\rangle \right) \\
&+ y_2 \cdot \frac{1}{2\sqrt{3}} \left(-3i |00\rangle + i |11\rangle + |01\rangle + |10\rangle \right)
\end{aligned} \tag{20}$$

For the convenience of calculation, denote $y_1 = r_1$, $y_2 = r_2 \cdot e^{i\varphi}$, here r_1, r_2 are positive numbers satisfying the relation $r_1^2 + r_2^2 = 1$, and φ is a real number.

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Direct calculation gives

$$
\begin{aligned}
\sigma_A = & \left(\frac{1}{2} + \frac{1}{3}r_2^2 + \frac{1}{\sqrt{3}}r_1 r_2 \sin\varphi \right) |0\rangle \langle 0| \\
& + \left(\frac{1}{2} - \frac{1}{3}r_2^2 - \frac{1}{\sqrt{3}}r_1 r_2 \sin\varphi \right) |1\rangle \langle 1| \\
& + \left(-\frac{1}{3}i \cdot r_2^2 + \frac{1}{\sqrt{3}}r_1 r_2 e^{i\varphi} \right) |0\rangle \langle 1| \\
& + \left(\frac{1}{3}i \cdot r_2^2 + \frac{1}{\sqrt{3}}r_1 r_2 e^{-i\varphi} \right) |1\rangle \langle 0|
\end{aligned}
\tag{21}
$$

The relation $\det(\sigma_A - I) = 0$ requires that

$$
r_2^2 = \frac{3\left(1 + \sin^2\varphi\right) \pm 3 \cdot \sqrt{\sin^4\varphi - \sin^2\varphi}}{2 + 6\sin^2\varphi}
\tag{22}
$$

Since r_2 is positive, we have

$$
\sin^4\varphi - \sin^2\varphi \geq 0
\tag{23}
$$

i.e.,

$$
\sin^2\varphi = 1
\tag{24}
$$

Here another solution $\sin^2\varphi = 0$ is not proper since $0 \leq r_2 \leq 1$. So we get

$$
\varphi = \pm\frac{\pi}{2}
\tag{25}
$$

and

$$
r_2 = \frac{\sqrt{3}}{2}
$$
$$
r_1 = \frac{1}{2}
$$

Therefore we get two (and only two) different vector solutions:

$$
\overrightarrow{y}^{(1)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}i \right)
$$

$$
\overrightarrow{y}^{(2)} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}i \right)
$$

In order to make the matrix

$$
M = \begin{pmatrix} \sqrt{p} & \sqrt{p}i \\ \sqrt{1-p} & -\sqrt{1-p}i \end{pmatrix}
\tag{26}
$$

left-unitary (also unitary in this case), there must be

$$
p_1 = p_2 = \frac{1}{2}
\tag{27}
$$

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And we have

$$
\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{pmatrix} \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}}\left(|\phi^+\rangle - i\,|\psi^+\rangle\right) \\ \frac{\sqrt{3}}{2}\frac{1}{2\sqrt{3}}\left(-3i\,|00\rangle + i\,|11\rangle + |01\rangle + |10\rangle\right) \end{pmatrix}
$$
$$
= \begin{pmatrix} \frac{1}{\sqrt{2}}\,|00\rangle \\ -\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right)\right] \end{pmatrix}
$$

That is to say, the bipartite state given in Eq. (19) is separable, and it can be rewritten as

$$
\rho_{AB} = \frac{1}{2}\,|0\rangle_A\,\langle 0| \otimes |0\rangle_B\,\langle 0| + \frac{1}{2}\,|\alpha\rangle_A\,\langle\alpha| \otimes |\alpha\rangle_B\,\langle\alpha| \tag{28}
$$

where $|\alpha\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right)$.

(3). Let us look at another example. The state of two qutrit systems is given by

$$
\rho_{AB} = \lambda \left[\frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)\right]\left[\frac{1}{\sqrt{3}}\left(\langle 00| + \langle 11| + \langle 22|\right)\right]
$$
$$
+ (1-\lambda)\left[\frac{1}{\sqrt{3}}\left(|01\rangle + |12\rangle + |20\rangle\right)\right]
$$
$$
\cdot \left[\frac{1}{\sqrt{3}}\left(\langle 01| + \langle 12| + \langle 20|\right)\right] \tag{29}
$$

Set

$$
|\Psi\rangle \equiv y_1 \cdot \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)
$$
$$
+ y_2 \cdot \frac{1}{\sqrt{3}}\left(|01\rangle + |12\rangle + |20\rangle\right) \tag{30}
$$

As before, denote $y_1 = r_1$, $y_2 = r_2 \cdot e^{i\varphi}$, here r_1, r_2 are positive numbers satisfying the relation $r_1^2 + r_2^2 = 1$, and φ is a real number.

Direct calculation gives

$$
\sigma_A = \frac{1}{3}\begin{pmatrix} 1 & r_1 r_2 e^{-i\varphi} & r_1 r_2 \\ r_1 r_2 & 1 & r_1 r_2 e^{-i\varphi} \\ r_1 r_2 e^{-i\varphi} & r_1 r_2 & 1 \end{pmatrix} \tag{31}
$$

The condition $\det(\sigma_A - I) = 0$ requires

$$
6 r_1^2 r_2^2 e^{-i\varphi} + r_1^3 r_2^3\left(1 + e^{-3i\varphi}\right) = 8 \tag{32}
$$

which is obviously impossible since $r_1 r_2 \le \frac{1}{\sqrt{2}}$. So there is no solution of Eqs. (3).

Thus we conclude that the bipartite qutrit state given by Eq. (29) is always entangled. In this example, the same result will be obtained if we use the rank separability criterion derived by Horodecki et al. [17], since the total density matrix has rank 2 while the reduced density matrices have ranks 3.

In conclusion, we have provided a necessary and sufficient condition for the separability of multipartite states. The key procedure of our criterion is to solve a set of equations, these equations can be

solved analytically if the density matrix of the given multipartite state has few nonzero eigenvalues, while numerical approach is always possible, in this sense, our criterion is operational.

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